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MOTION OF A THIN BLANKET OF HEAVY VISCOUS FLUID ON A ROTATING PLANET IN A CIRCULAR ORBIT

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The system of Navier–Stokes equations which define the motion of a heavy viscous incompressible fluid in the blanket covering a rotating sphere whose center moves along a circle is solved with allowance for transport and Coriolis inertia forces. The derived solution is valid on the assumption that the sphere radius is considerably greater than the mean thickness of the fluid blanket and that either the axis of sphere rotation diverges slightly from the normal to the orbit plane, or that the angular rotation velocity Ω of the sphere is considerably greater than the angular rotation velocity ω of the sphere center about the orbit center. Existence of lattitudes on the sphere at which separation of fluid from its free surface is possible is established. Relation between Ω and ω at which intensive meridional currents from the equator to the poles are possible in the fluid blanket, is determined. The particular case of $\omega = 0$ of flow in the fluid blanket induced by the rotation of the sphere about its axis is analyzed.

1. Let us consider a sphere of radius a rotating about one of its diameters at constant angular velocity Ω and its center describing a circle of radius R at constant angular velocity ω relative to the circle center O_1. We assume that the angle ψ between $\mathbf{\Omega}$ and $\boldsymbol{\omega}$ is constant. We use a Cartesian coordinate system whose origin is at the sphere center O, the Ox- and Oz-axes coincide, respectively, with the intersection line of the equatorial and orbit planes and of vector $\mathbf{\Omega}$; the Oy-axis lies in the equatorial plane (Fig. 1, a). The Ox-axis orientation remains unchanged during the motion of the sphere on its orbit, i.e. its motion is translational. We denote by $\mathbf{R}_0$ the initial position of vector $\mathbf{R}$ drawn from the orbit center O_1 to the sphere center O, which is parallel to the Ox-axis and faces in the opposite direction. Angle χ between the Ox-axis and vector $(-\mathbf{R})$ is, then, equal to the angle of turn of vector $\mathbf{R}$ from the initial position $\mathbf{R}_0$ (Fig. 1, b). Hence

$$\chi = \omega t', \ \omega_x = 0, \ \omega_y = -\omega \sin\psi, \ \omega_z = \omega\cos\psi \tag{1.1}$$
$$R_x = -R\cos\chi, \ R_y = -R\sin\chi\cos\psi, \ R_z = -R\sin\chi\sin\psi$$

We denote by g the acceleration of gravity of the field created by the sphere, and assume that the sphere is covered by a blanket of heavy viscous incompressible fluid. We disregard tidal forces and determine the velocity field $\mathbf{v}$ and the hydrodynamic pressure p' in the fluid in its relative motion around the sphere. Since the rotation around center O_1 at angular velocity ω is the transport motion, the Coriolis acceleration is $2(\boldsymbol{\omega}\times\mathbf{v})$, and the transport acceleration is defined by

$$\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{R}+\mathbf{r})] = \nabla\left[\frac{(\boldsymbol{\omega}\cdot\mathbf{r})^2}{2} - \frac{\omega^2 r^2}{2} - \omega^2(\mathbf{r}\cdot\mathbf{R})\right]$$

where $\mathbf{r}$ is the radius vector from the sphere center to a point of fluid. This means that the Navier–Stokes equations of the fluid relative motion are

$$\partial \mathbf{v}/\partial t' + (\mathbf{v}\nabla)\mathbf{v} + \nabla [p'/\rho + gr + (\omega\cdot\mathbf{r})^2/2 - \omega^2 r^2/2 - \omega^2(\mathbf{r}\cdot\mathbf{R})] + 2(\omega\times\mathbf{v}) = \nu\nabla^2\mathbf{v}, \quad \nabla\mathbf{v} = 0 \tag{1.2}$$

These equations must be solved on the assumption that the fluid adheres to the surface of the sphere and that normal and tangent stresses are absent at the unknown free surface. Since the transport acceleration,

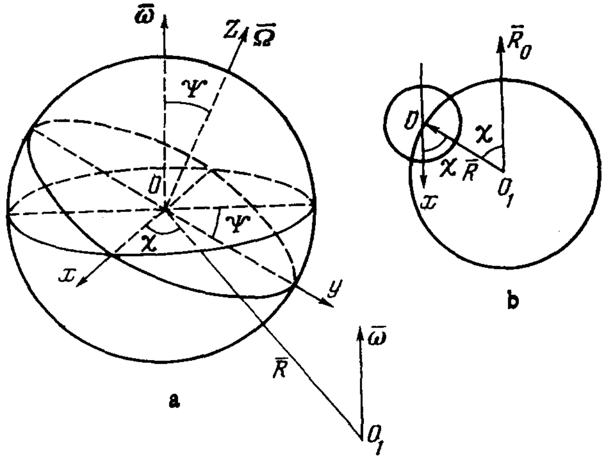

Fig. 1

an external force appearing in (1.2), is of period $2\pi/\omega$, the general solution of the considered problem must be of the same period with respect to t' for any arbitrary ω. The exception is presented by the eigenvalues of vector ω for which bifurcation of solutions of the homogeneous boundary value problem (1.2) is possible. Such values of ω are not considered here. The analysis is limited to purely periodic solutions.

Problems similar to the one considered here are found in oceanography [1], meteorology [2], and the theory of planets [3].

We introduce spherical coordinates r, θ, φ with their pole at the sphere center O. Latitude θ is measured from the sphere rotation axis Ω and the longitude φ from the half plane xOz. In this system the condition of adhesion and absence of stresses at the free surface $r = \zeta'(\theta, \varphi, t')$ are of the form

$$v_r = v_\theta = 0, \quad v_\varphi = \Omega a \sin\theta \quad \text{for} \quad r = a \tag{1.3}$$

$$p' = 2\rho\nu \frac{\partial v_r}{\partial r}, \quad \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} = \frac{v_\theta}{r}$$

$$\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \varphi} = \frac{v_\varphi}{r} \quad \text{for} \quad r = \zeta'$$

Function $\zeta'(\theta, \varphi, t')$ which determines the free surface satisfies the equation

$$\frac{\partial \zeta'}{\partial t'} = v_r - \frac{v_\theta}{r}\frac{\partial \zeta'}{\partial \theta} - \frac{v_\varphi}{r\sin\theta}\frac{\partial \zeta'}{\partial \varphi}\bigg|_{r=\zeta'} \tag{1.4}$$

We introduce the characteristic length $L = \delta$ equal to the thickness of the fluid blanket of the stationary sphere, and the characteristic velocity $V = \nu\delta^{-1}$. We then obtain such fundamental dimensionless characteristics of flow as: the Reynolds, the Froude, and the Rossby numbers

$$VL\nu^{-1} = 1, \quad V^2 (gL)^{-1} = \nu^2 g^{-1}\delta^{-3}, \quad V (2\Omega L)^{-1} = \nu\delta^{-2} (2\Omega)^{-1}$$

We further assume that

$$v_\varphi = \Omega r \sin\theta + w_\varphi \tag{1.5}$$

and taking into account (1.1), write (1.2) in spherical coordinates. We then substitute expression (1.5) into (1.3) and (1.4) and pass in the obtained equations and boundary conditions to the dimensionless variables

$$\varepsilon = \frac{\delta}{a}, \quad r = a(1 + \varepsilon x), \quad t' = \frac{\delta^2 t}{\nu}, \quad \lambda = \frac{\Omega\delta^2}{\nu}, \quad \zeta' = a(1 + \varepsilon\zeta) \tag{1.6}$$

$$v_r = \frac{\varepsilon\nu}{\delta} u, \quad v_\theta = \frac{\nu}{\delta} v, \quad w_\varphi = \frac{\nu}{\delta} w, \quad p' = \varepsilon\frac{\rho\nu^2 a}{\delta^3} p, \quad \alpha = \frac{\omega\delta^2}{\nu}$$

For the determination of u, v, w, p, and ζ we now have the following equations and boundary conditions:

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - \lambda\frac{\partial v}{\partial\varphi} + 2\lambda w\left(\cos\theta + \frac{\omega}{\Omega}\varkappa\right) - \frac{\partial q}{\partial\theta} + \tag{1.7}$$

$$2\frac{\Omega\omega a\delta^3}{\nu^2}\varkappa\sin\theta = \varepsilon f_1(u, v, w, p, \varepsilon), \quad \frac{\partial q}{\partial x} = \varepsilon f_3(u, v, w, \varepsilon)$$

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial t} - \lambda\frac{\partial w}{\partial\varphi} - 2\lambda v\left(\cos\theta + \frac{\omega}{\Omega}\varkappa\right) - \frac{1}{\sin\theta}\frac{\partial q}{\partial\varphi} = \varepsilon f_2(u, v, w, p, \varepsilon)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi} + v\operatorname{ctg}\theta = \varepsilon f_4(u, v, w, \varepsilon)$$

$$u = v = w = 0 \quad \text{for } x = 0$$

$$p = \varepsilon f_5(u), \quad \frac{\partial v}{\partial x} = \varepsilon f_6(u, v, \varepsilon), \quad \frac{\partial w}{\partial x} = \varepsilon f_7(u, w, \varepsilon) \quad \text{for } x = \zeta$$

$$\frac{\partial\zeta}{\partial t} + \lambda\frac{\partial\zeta}{\partial\varphi} = \varepsilon f_8(u, v, w, \zeta, \varepsilon)$$

where

$$q = \varepsilon p + \frac{g\delta^3}{\nu^2}(1 + \varepsilon x) - \frac{n}{2}(1 + \varepsilon x)^2\sin^2\theta + m(1 + \varepsilon x)\times \tag{1.8}$$

$$\Big[\sin\theta\cos\varphi\cos\alpha t + (\cos\psi\sin\theta\sin\varphi + \sin\psi\cos\theta)\sin\alpha t +$$

$$\frac{a}{2R}(1 + \varepsilon x)(\varkappa^2 - 1)\Big]$$

$$\varkappa = \cos\psi\cos\theta - \sin\psi\sin\theta\sin\varphi$$

$$m = \frac{\omega^2\delta^3 R}{\nu^2}, \quad n = \frac{\Omega^2\delta^3 a}{\nu^2} \tag{1.9}$$

and $f_1, f_2, \ldots, f_8$ are differentiation operators of an order not higher than the second. Since they are functions of ε, they are bounded for $\varepsilon = 0$. Hence the solution of problem (1.7) for small ε can be sought in the form of series in positive powers of ε; for the zero terms of these series it is necessary to use equations and boundary conditions that are obtained from (1.7) for $\varepsilon = 0$. For the determination of the free surface we have the equation

$$\frac{\partial \zeta}{\partial t} + \lambda \frac{\partial \zeta}{\partial \varphi} = 0$$

which is periodic in t and φ and whose solution is $\zeta = \text{const}$. Then it follows from (1.6) that $\zeta = 1$, and (1.7) yields equations and boundary conditions for p and q in the zero approximation

$$\frac{\partial q}{\partial x} = 0, \quad p|_{x=1} = 0 \tag{1.10}$$

Setting in (1.8) $x = 1$ and using (1.10), we obtain the formula for q which is valid for all x, since q is independent of x, and p for $x = 1$ is zero. The obtained expression for q is introduced in (1.8), and the resulting equation is solved for the dimensionless hydrodynamic pressure in the fluid blanket, yielding

$$p = \frac{g\delta^3}{\nu^2}(1-x) - n(1-x)\sin^2\theta + m(1-x)\Big[\sin\theta\cos\varphi\cos\alpha t +$$
$$(\cos\psi\sin\theta\sin\varphi + \sin\psi\cos\theta)\sin\alpha t + \frac{a}{R}(\varkappa^2 - 1)\Big] + O(\varepsilon)$$

Let us further assume that the orbit radius is considerably greater than the sphere radius, and that either the axis of rotation of the sphere is slightly divergent from the normal to the orbit plane, or that the frequency of the sphere rotation about its axis considerably exceeds the frequency of rotation of the sphere about the orbit axis, i.e.

$$a \ll R, \quad \left|\frac{\omega}{\Omega}\sin\psi\right| \ll 1 \tag{1.11}$$

Substituting on these assumptions (1.8) into (1.7) and setting $\varepsilon = 0$, we obtain the equations and boundary conditions of the zero approximation for u, v, and w

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - \lambda\frac{\partial v}{\partial \varphi} + \eta w = -n\left(1 + \frac{2\omega}{\Omega}\cos\psi\right)\frac{\sin 2\theta}{2} + \tag{1.12}$$
$$\frac{im}{2}(e^{i\alpha t} - e^{-i\alpha t})\sin\psi\sin\theta + \frac{m}{2}\cos\theta\sin^2\frac{\psi}{2}[e^{i(\alpha t+\varphi)} + e^{-i(\alpha t+\varphi)}] +$$
$$\frac{m}{2}\cos\theta\cos^2\frac{\psi}{2}[e^{i(\alpha t-\varphi)} + e^{-i(\alpha t-\varphi)}]$$

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial t} - \lambda\frac{\partial w}{\partial \varphi} - \eta v = \frac{im}{2}\sin^2\frac{\psi}{2}[e^{i(\alpha t+\varphi)} - e^{-i(\alpha t+\varphi)}] -$$
$$\frac{im}{2}\cos^2\frac{\psi}{2}[e^{i(\alpha t-\varphi)} - e^{-i(\alpha t-\varphi)}]$$

$$\eta = 2\lambda\left(1 + \frac{\omega}{\Omega}\cos\psi\right)\cos\theta$$

$$v = w = 0 \quad \text{for} \quad x = 0, \quad \partial v/\partial x = \partial w/\partial x = 0 \quad \text{for} \quad x = 1$$

Having determined v and w, for the dimensionless radial velocity we obtain the formula

$$u = -\int_0^x \left(\frac{\partial v}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi} + v\operatorname{ctg}\theta\right)dx$$

The solution of system (1.12) is of the form

$$v = v_0 + 2\,\mathrm{Re}\,[v_1 e^{i\alpha t} + v_2 e^{i(\alpha t+\varphi)} + v_3 e^{i(\alpha t-\varphi)}]$$
$$w = w_0 + 2\,\mathrm{Re}\,[w_1 e^{i\alpha t} + w_2 e^{i(\alpha t+\varphi)} + w_3 e^{i(\alpha t-\varphi)}] \qquad (1.13)$$

where v_j and w_j $(j = 0, 1, 2, 3)$ satisfy a system of equations and boundary conditions of the form

$$\frac{d^2V}{dx^2} - i\beta V + \eta W = A, \quad \frac{d^2W}{dx^2} - i\beta W - \eta V = B$$

$$V = W = 0 \quad \text{for } x = 0, \quad \frac{dV}{dx} = \frac{dW}{dx} = 0 \quad \text{for } x = 1$$

where β takes one of the following values: $0, \alpha, \alpha \pm \lambda$, and A and B depend only on θ and for each β are determined by right-hand sides of (1.12) after the substitution of (1.13) into the left-hand sides and equating the coefficients at equal exponential functions.

The system (1.12) of ordinary differential equations and boundary conditions with constant coefficients and free terms has the solution

$$V = AF(\beta, x, |\eta|) + BG(\beta, x, \eta), \quad W = -AG(\beta, x, \eta) + BF(\beta, x, |\eta|) \qquad (1.14)$$

$$F(\beta, x, |\eta|) = i\left[\frac{\beta}{\beta^2-\eta^2} - \frac{\mathrm{ch}\,\sigma_1(1-x)}{2(\beta+|\eta|)\,\mathrm{ch}\,\sigma_1} - \frac{\mathrm{ch}\,\sigma_3(1-x)}{2(\beta-|\eta|)\,\mathrm{ch}\,\sigma_3}\right]$$

$$G(\beta, x, \eta) = \left[\frac{|\eta|}{\beta^2-\eta^2} + \frac{\mathrm{ch}\,\sigma_1(1-x)}{2(\beta+|\eta|)\,\mathrm{ch}\,\sigma_1} - \frac{\mathrm{ch}\,\sigma_3(1-x)}{2(\beta-|\eta|)\,\mathrm{ch}\,\sigma_3}\right]\mathrm{sgn}\,\eta$$

$$\sigma_{1,3} = \sqrt{|\beta \pm |\eta||}\exp\left[\frac{i\pi}{4}\,\mathrm{sgn}\,(\beta \pm |\eta|)\right]$$

Note here also the relationships

$$S(|\eta|, x) = 1 - \frac{\mathrm{ch}\,[e^{1/4\pi i}(1-x)\sqrt{|\eta|}]}{\mathrm{ch}\,(e^{1/4\pi i}\sqrt{|\eta|})} \qquad (1.15)$$

$$S^*(|\beta|, x) = \begin{cases} S(|\beta|, x), & \beta > 0 \\ \bar{S}(|\beta|, x), & \beta < 0 \end{cases}$$

$$\lim_{|\eta|\to|\beta|} F(\beta, x, |\eta|) = \frac{i}{4\beta} S^*(2|\beta|, x) - \frac{x(2-x)}{4}$$

$$\lim_{|\eta|\to 0} \frac{1}{|\eta|} S(|\eta|, x) = \frac{ix(2-x)}{2}$$

$$\lim_{|\eta|\to|\beta|} G(\beta, x, \eta) = -\frac{\mathrm{sgn}\,\eta}{4\beta} S^*(2|\beta|, x) + \frac{ix(2-x)}{4}\,\mathrm{sgn}\,\eta\cdot\mathrm{sgn}\,\beta$$

$$\lim_{\beta\to\pm 0} \frac{1}{|\beta|} S^*(|\beta|, x) = \pm\frac{ix(2-x)}{2}$$

where the upper dash denotes quantities complex conjugate to S.

The formulas for functions v_j and w_j that appear in (1.13) and are obtained from (1.14) for corresponding values of β, A, and B are

$$v_0 = \frac{n}{|\eta|}\left(\frac{1}{2} + \frac{\omega}{\Omega}\cos\psi\right)\operatorname{Im} S(|\eta|, x)\sin 2\theta \tag{1.16}$$

$$w_0 = -\frac{n}{\eta}\left(\frac{1}{2} + \frac{\omega}{\Omega}\cos\psi\right)\operatorname{Re} S(|\eta|, x)\sin 2\theta$$

$$v_1 = \frac{im}{2}F(\alpha, x, |\eta|)\sin\psi\sin\theta, \quad w_1 = -\frac{im}{2}G(\alpha, x, \eta)\sin\psi\sin\theta$$

$$v_2 = \frac{m}{2}\sin^2\frac{\psi}{2}(F_+\cos\theta + iG_+), \quad w_2 = \frac{m}{2}\sin^2\frac{\psi}{2}(-G_+\cos\theta + iF_+)$$

$$v_3 = \frac{m}{2}\cos^2\frac{\psi}{2}(F_-\cos\theta - iG_-), \quad w_3 = -\frac{m}{2}\cos^2\frac{\psi}{2}(G_-\cos\theta + iF_-)$$

$$F_\pm = F(\alpha \pm \lambda, x, |\eta|), \quad G_\pm = G(\alpha \pm \lambda, x, \eta)$$

2. Leaving aside a complete analysis of formulas (1.13) and (1.16), we shall examine the case of $\alpha \to \infty$ and $\lambda \to \infty$, when (1.14)–(1.16) imply that the ratios v_0/n, w_0/n, v_j/m and w_j/m $(j = 1, 2, 3)$ tend to vanish at all latitudes θ, except those that are determined by the conditions $|\eta| = 0$, $|\alpha|$, and $|\alpha \pm \lambda|$. We shall consider each of these cases separately.

In the first case, $|\eta| = 0$, formulas (1.13) by virtue of (1.16) and (1.15) assume the form

$$\begin{aligned} v = {} & \frac{n}{2}\left(\frac{1}{2} + \frac{\omega}{\Omega}\cos\psi\right)x(2-x)\sin 2\theta - \\ & \frac{m}{\alpha}\operatorname{Re}[S^*(|\alpha|, x)e^{i\omega t'}]\sin\psi\sin\theta + \\ & \frac{m\sin^2 {}^1\!/_2\psi}{\alpha+\lambda}\operatorname{Re}[iS^*(|\alpha+\lambda|, x)e^{i(\omega t'+\varphi)}]\cos\theta + \\ & \frac{m\cos^2 {}^1\!/_2\psi}{\alpha-\lambda}\operatorname{Re}[iS^*(|\alpha-\lambda|, x)e^{i(\omega t'-\varphi)}]\cos\theta \\ w = {} & -\frac{m\sin^2 {}^1\!/_2\psi}{\alpha+\lambda}\operatorname{Re}[S^*(|\alpha+\lambda|, x)e^{i(\omega t'+\varphi)}] + \\ & \frac{m\cos^2 {}^1\!/_2\psi}{\alpha-\lambda}\operatorname{Re}[S^*(|\alpha-\lambda|, x)e^{i(\omega t'-\varphi)}] \quad (|\eta| = 0) \end{aligned} \tag{2.1}$$

It follows from (1.12) that the equality $|\eta| = 0$ is possible in two cases.

a) $\theta = \pi/2$ (latitude of the equator). The first of formulas (2.1) yields

$$v = -\frac{m}{\alpha}\operatorname{Re}[S^*(|\alpha|, x)e^{i\omega t'}]\sin\psi \quad (\theta = \pi/2)$$

and the second remains unchanged. If the angular velocity of the sphere about its axis is equal to that of the sphere motion along the orbit $(\alpha = \lambda)$, then, as implied by (2.1) and (1.15),

$$w = -\frac{1}{2}mx(2-x)\cos^2\frac{\psi}{2}\sin(\omega t' - \varphi) + O\left(\frac{m}{\lambda}\right)$$
$$(\theta = \pi/2, \ \alpha = \lambda \to \infty)$$

In dimensional form that velocity, with allowance for (1.6) and (1.9), is of the form

$$w_\varphi = -\frac{\omega^2\delta^2 R}{2\nu}\cos^2\frac{\psi}{2}\sin(\omega t' - \varphi) \tag{2.2}$$

$$(\theta = \pi/2,\ \omega\delta^2/\nu = \Omega\delta^2/\nu \to \infty,\ x = 1)$$

If $\omega = -\Omega$, then in virtue of (2.1) we have a similar situation:

$$w_\varphi = \frac{\omega^2\delta^2 R}{2\nu}\sin^2\frac{\psi}{2}\sin(\omega t' + \varphi) \tag{2.3}$$

$$(\theta = \pi/2,\ -\Omega\delta^2/\nu = \omega\delta^2/\nu \to \infty,\ x = 1)$$

Thus, when $\alpha \neq \pm\lambda$, $\lambda \to \infty$, $\Omega \neq -\omega\cos\psi$, $|\eta| \neq |\alpha|$, and $|\alpha \pm \lambda|$, the ratios v/m and w/m tend to vanish at all latitudes. If $\alpha = \pm\lambda$, these ratios tend to vanish at all latitudes, except in the equator neighborhood where the transversal velocity at the free surface is determined by formulas (2.2) or (2.3) with allowance for (1.5). For a fairly large thickness δ of the fluid blanket the transversal velocities (2.2) and (2.3) in the equator neighborhood can be arbitrarily high. This means that under conditions considered here the sphere is incapable of retaining at its surface a blanket of arbitrary thickness.

b) $\Omega = -\omega\cos\psi$. In this case it is necessary to specify $\psi \sim 0$, i.e. that the sphere axis of rotation is to be slightly divergent from the normal to the orbit plane, if condition (1.11) is to be satisfied. On these assumptions formulas (2.1) are of the form

$$v = -\frac{n}{2}x(2 - x)\sin\theta\cos\theta + O\left(\frac{n}{\lambda}\right)$$

$$w = O\left(\frac{n}{\lambda}\right) \qquad (\Omega = -\omega\cos\psi,\ \lambda \to \infty)$$

or in dimensional form at the free surface

$$v_\theta = -\frac{\Omega^2\delta^2 a}{2\nu}\sin\theta\cos\theta \quad (\Omega = -\omega\cos\psi,\ \Omega\delta^2/\nu \to \infty,\ x = 1) \tag{2.4}$$

This means that, when the sphere rotation about its axis is in opposite direction to that of its rotation about the orbit axis and $\Omega = -\omega\cos\psi$, then intensive meridional flows from the equator to the poles at velocity (2.4) are observed.

In the second case

$$|\eta| = |\alpha|, \quad \theta_1 = \arccos\left|\frac{\omega}{2(\Omega + \omega\cos\psi)}\right|, \quad \theta_2 = \pi - \theta_1 \tag{2.5}$$

formulas (1.13) – (1.16) yield for considerable α and λ with allowance for (1.6) and (1.9) the following dimensional relationships:

$$v_\theta = \frac{\omega^2\delta^2 R}{4\nu}\sin\psi\sin\omega t'\sin\theta \tag{2.6}$$

$$w_\varphi = \frac{\omega^2\delta^2 R}{4\nu}\sin\psi\cos\omega t'\sin\theta\,\mathrm{sgn}\,\eta\,\mathrm{sgn}\,\alpha \qquad (|\eta| = |\alpha|,\ x = 1)$$

Furthermore, if $\omega = \pm\Omega/2$, then at latitudes (2.5) terms of the same order and of form

$$\frac{\omega^2\delta^2 R}{2\nu}(1 \pm \cos\psi)(1 \pm \cos\theta)\,{}^{\cos}_{\sin}(\omega t' \pm \varphi)$$

are added in (2.6).

In the third case

$$|\eta| = |\alpha \pm \lambda|, \quad \theta_3 = \arccos\left|\frac{\Omega \pm \omega}{2(\Omega + \omega\cos\psi)}\right|, \quad \theta_4 = \pi - \theta_3 \qquad (2.7)$$

we obtain similarly to (2.6)

$$v_\theta = -\frac{\omega^2\delta^2 R}{2\nu}(1 \mp \cos\psi)[\cos\theta \pm \operatorname{sgn}\eta\operatorname{sgn}(\alpha \pm \lambda)]\cos(\omega t' + \varphi) \qquad (2.8)$$

$$w_\varphi = \frac{\omega^2\delta^2 R}{2\nu}(1 \mp \cos\psi)[\cos\theta\operatorname{sgn}\eta\operatorname{sgn}(\alpha \pm \lambda) \pm 1]\sin(\omega t' + \varphi)$$

$$(|\eta| = |\alpha \pm \lambda|,\ x = 1)$$

and if $\omega = \pm\Omega$ or $\omega = \pm\Omega/2$, terms of the same order are added at latitudes (2.7) in (2.8).

It follows from (2.6) and (2.8) that for fairly large δ, R, and ω, as well as for fairly small ν separation of fluid from the free surface must occur at latidues (2.5) and (2.7) at velocities (2.6) and (2.8), respectively, independently of the manner of increase of α and λ.

Note that the possibility of fluid escape from the equator is, generally speaking, obvious, and since in the equality (1.5) this is determined by the term Ωa, is not affected by viscosity. However velocities (2.2), (2.3), (2.6), and (2.8) at corresponding latitudes can exceed Ωa. In such cases separation of the blanket fluid is determined not only by viscosity but, also, thickness δ, the orbit radius R, the angular velocity ω of rotation of the sphere about the orbit axis.

3. In conclusion we consider the simple limit case of $\omega = 0$ which relates to the motion of fluid induced by the rotation of the sphere about a stationary axis. In this case $m = 0$ and formulas (1.13) and (1.16) yield

$$v = \frac{n\sin\theta}{2\lambda}\left[1 - \frac{1}{\Delta}(\operatorname{ch}\xi y\cos\xi y\operatorname{ch}\xi\cos\xi + \operatorname{sh}\xi y\sin\xi y\operatorname{sh}\xi\sin\xi)\right] \qquad (3.1)$$

$$w = \frac{n\sin\theta}{2\lambda\Delta}(\operatorname{ch}\xi y\cos\xi y\operatorname{sh}\xi\sin\xi - \operatorname{sh}\xi y\sin\xi y\operatorname{ch}\xi\cos\xi)$$

$$\Delta = \operatorname{ch}^2\xi\cos^2\xi + \operatorname{sh}^2\xi\sin^2\xi, \quad y = 1 - x, \quad \xi = \sqrt{\lambda\cos\theta}$$

Since for $\omega = 0$ the flow of fluid is steady and independent of angle φ, and symmetric about the equatorial plane, it is sufficient to consider the values $0 \leqslant \theta \leqslant \pi/2$. For radial velocity we have

$$u = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\int_y^1 v\sin\theta\,dy \qquad (3.2)$$

which together with boundary conditions (1.12) implies that u, v, and w attain their maximum absolute values at the free surface. Hence it is sufficient to investigate formulas (3.1) and (3.2) for $y = 0$

$$u = -\frac{n\cos\theta}{\lambda}\left(1 - \frac{\operatorname{sh} 2\xi + \sin 2\xi}{4\Delta}\right) - \tag{3.3}$$

$$\frac{n\sin^2\theta}{8\xi^2\Delta}\left(\operatorname{ch} 2\xi + \cos 2\xi - \frac{\operatorname{sh} 2\xi + \sin 2\xi}{2\xi} - \frac{\operatorname{sh}^2 2\xi - \sin^2 2\xi}{2\Delta}\right)$$

$$v = \frac{n\sin\theta}{2\lambda}\left(1 - \frac{\operatorname{ch}\xi\cos\xi}{\Delta}\right), \quad w = \frac{n\sin\theta\operatorname{sh}\xi\sin\xi}{2\lambda\Delta}$$

Since in the equatorial region, $\theta \to \pi/2$, ξ is a small parameter, formulas (3.3) are of the form

$$u = \frac{8n\lambda}{15}\cos\theta + O(\cos^3\theta), \quad w = \frac{n}{2}\cos\theta + O(\cos^2\theta)$$

$$v = \frac{5n\lambda}{12}\cos^2\theta + O(\cos^3\theta) \quad (x = 1,\ \theta \to \pi/2)$$

Although at the equator itself $u = v = w = 0$, the radial and the transverse velocities in the equator neighborhood decrease slower than the meridional velocity, there is in equatorial latitudes a flow of fluid from the depth to the surface.

We restrict our analysis at some distance from the equator to the case of $\lambda \to \infty$, when also $\xi \to \infty$, and formulas (3.3) yield

$$u = -\frac{n\cos\theta}{\lambda} + O(n\lambda^{-3/2}), \quad v = \frac{n\sin\theta}{2\lambda}[1 + O(e^{-\sqrt{\lambda\cos\theta}})]$$

$$w = \frac{n}{\lambda}e^{-\sqrt{\lambda\cos\theta}}\sin\theta\sin(\sqrt{\lambda\cos\theta})[1 + O(e^{-2\sqrt{\lambda\cos\theta}})]$$

$$(x = 1,\ \theta \not\to \pi/2)$$

At the pole

$$v = w = 0 \quad (\theta = 0), \quad u = -\frac{n}{\lambda} + O(n\lambda^{-3/2}) \quad (x = 1,\ \theta = 0)$$

i.e. radial flows of fluid from the surface to the depth predominate at polar latitudes.

The obtained results may be applied to various problems of the theory of rotating fluids [4] and in the theory of planets which is at present the subject of considerable attention. For instance, the numerical modeling carried out in [5] reproduces the axsymmetric structure of Jupiter bands, which is in qualitative agreement with the conclusions obtained above for latitudes (2.5) and (2.7).

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